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Global Synchronization & Anti-Synchronization in N-Coupled Map Lattices

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Abstract By considering a symmetric N-dimensional map which possesses invariant measure in its diagonal and anti-diagonal invariant sub-manifolds, we have been able to propose an N-coupled map which possesses invariant measure in synchronized or anti-synchronized states. Then chaotic synchronization and anti-synchronization are investigated in the introduced model. We have calculated Kolmogrov–Sinai entropy and Lyapunov exponent as another tool to study the stability of N-coupled map in synchronized and anti-synchronization states.

Keywords Synchronization · Anti-synchronization · Invariant measure · Kolmogrov–Sinai entropy · Lyapunov exponent

1 Introduction

How to simulate of the natural phenomena has been a major field of research in nowadays world meanwhile coupled map lattices has been a favored paradigm for studying fundamental questions in spatially extended dynamical systems [1–5]. Synchronization of two identical chaotic dynamical systems has received much attention in the mathematics and physics literature in the last few years [6]. In this field, the key concept of complete synchronization refers to a state where the trajectories of dynamical systems approach each other [7–9]. By consideration a symmetric N-dimensional map possessing invariant measure in its diagonal and anti-diagonal invariant sub-manifolds, we have been able to introduce the N-coupled

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map which possesses invariant measure at synchronized or anti-synchronized states (by anti-synchronization we mean the disappearance of the sum of relevant variables). In both synchronization and anti-synchronization states, an analytic expression is given for the invariant measure of the model. Then using this invariant measure, its Kolmogorov–Sinai (KS) entropy has been calculated and is compared to its Lyapunov exponents. It is a simple process to apply our criteria to the system being studied.

2 N-coupled Maps

Coupled map lattices are arrays of states whose values are continuous, usually within a unit interval, or discrete space and time [10]. The N-coupled map dynamical system can be considered as an N-dimensional dynamical map defined as:

$$\Phi(x_1,\dots,x_N)=\begin{cases} X_1=F(x_1,x_2,x_3,\dots,x_N),\\ X_2=F(x_2,x_3,\dots,x_N,x_1),\\ \vdots\\ X_N=F(x_N,x_1,\dots,x_{N-1}).\end{cases}\tag{1}$$

By an appropriate choice of the function $F(x_1,x_2,\dots,x_N)$, we may have an N-dimensional dynamical system with the property of possessing invariant measure in invariant submanifolds $x_1=x_2=\dots=x_N=x$. It can be easily verified so that this state will be $X_1=X_2=\dots=X_N=X$. Clearly the above-mentioned condition may not meet unless the one-dimensional measurable dynamical systems are implemented in the N-coupled maps. On the other hand, most linearly or non-linearly symmetric N-coupled maps can be considered as a symmetric N-dimensional map of the form given in (1). Following the underneath as the example:

$$\Phi=\begin{cases} X_1=[\epsilon_1(f_1(x_1))^P+\epsilon_2(f_2(x_2))^P+\dots+\epsilon_N(f_N(x_N))^P]^{\frac{1}{p}},\\ \vdots\\ X_N=[\epsilon_1(f_1(x_N))^P+\epsilon_2(f_2(x_1))^P+\dots+\epsilon_N(f_{N-1}(x_{N-1}))^P]^{\frac{1}{p}}\end{cases}\tag{2}$$

where, in general, p is an arbitrary parameter, ϵ the strength of coupling, and the functions $f_1(x_1), f_2(x_2),\dots, f_N(x_N)$ are N arbitrary one-dimensional maps. Obviously, by choosing $p=1$, we get ordinary linearly coupled maps. But, in order to have an N-dimensional dynamical system or accordance with the N-coupled map with the property of possessing an invariant measure at synchronized state, one needs to choose p as an arbitrary integer and the functions $f(x)$, as measurable dynamical system (see Appendix). Also in order to have the same condition at anti-synchronized state one needs to choose p as an odd integer and the functions $f(x)$, odd as well.

3 Stability Analysis

Chaos synchronization has potential applications in secure communication, information processing and pattern formation. Several types of synchronization may occur in coupled map systems, given the different phase regimes in which the individual and collective component maps can operate. Synchronization is one of the invariant manifolds of dynamical systems [11, 12].

3.1 Invariant Measure in Synchronized State

Dynamical systems, even apparently simple dynamical systems which are described by maps of an interval can display a rich variety of different asymptotic behaviors. On measure theoretical level these types of behaviors are describes by Sinai–Ruelle–Bowen (SRB). In addition the invariant measure describes statistically stationary states of the system [13]. The probability measure μ for the symmetric N-dimensional map $\Phi(x_1,\ldots,x_N)$ given in (1) fulfills the following formal Frobenius–Perron (FP) integral equation [14, 15]:

$$\begin{aligned}\mu(X_1,\ldots,X_N) = \int dx_1\cdots\int dx_N\delta(X_1 - F(x_1,x_2,\ldots,x_N))\\ \times\delta(X_2 - F(x_2,x_3,\ldots,x_1))\times\cdots\\ \times\delta(X_N - F(x_N,x_1,\ldots,x_{N-1}))\times\mu(x_1,x_2,\ldots,x_N),\end{aligned}$$

the corresponding FP equation can be written as:

$$\mu = \sum_{(x_{1,k},\ldots,x_{N,k})\in\Phi^{-1}} J(x_{1,k},\ldots,x_{N,k})\mu(x_{1,k},\ldots,x_{N,k}),\quad k=1,2,\ldots,\tilde{M} \tag{3}$$

where $x_{i,k}$ the roots of equations are defined in (1), for given X_i $(i=1,\ldots,N)$. The Jacobian $J(x_{1,k},\ldots,x_{N,k})$ is defined as:

$$J(x_{1,k},\ldots,x_{N,k}) = \left|\det\begin{pmatrix}\frac{\partial X_1}{\partial x_{1,k}} & \frac{\partial X_1}{\partial x_{2,k}} & \cdots & \frac{\partial X_1}{\partial x_{N,k}}\\ \frac{\partial X_2}{\partial x_{1,k}} & \frac{\partial X_2}{\partial x_{2,k}} & \cdots & \frac{\partial X_2}{\partial x_{N,k}}\\ \vdots & & & \\ \frac{\partial X_N}{\partial x_{1,k}} & \frac{\partial X_N}{\partial x_{2,k}} & \cdots & \frac{\partial X_N}{\partial x_{N,k}}\end{pmatrix}\right|. \tag{4}$$

By considering the one-dimensional maps $X=F(x,x,\ldots,x)$ with invariant measure $\mu(x)$, one can prove that, in synchronized states the invariant measure of invariant sub-manifolds $(x_{1,k}=\cdots=x_{N,k}=x \Longrightarrow X_1=\cdots=X_N=X)$ may take the following form:

$$\mu(x_1,\ldots,x_N) = \delta(x_2-x_1)\cdots\delta(x_N-x_1)\mu(x_1), \tag{5}$$

where $\mu(x_1)$ corresponds to the invariant measure of one-dimensional map $X=F(x,\ldots,x)$. We insert $\mu(x_1,\ldots,x_N)=\delta(x_2-x_1)\cdots\delta(x_N-x_1)\mu(x_1)$ in (3), and using the Dirac delta function, we get:

$$\begin{aligned}\sum_{x_{1,k_1}}\frac{\mu(x_{1,k_1})}{|\sum_{j=1}^N h_j(x_{1,k_1})|}\times\sum_{x_{1,k_1}}\frac{\delta(x_{1,k_1}-x_{2,k_2})}{|\sum_{j=1}^N\exp(\frac{2\pi i(j-1)}{N})h_j(x_{1,k_1})|}\times\cdots\\ \times\sum_{x_{1,k_N}}\frac{\delta(x_{1,k_1}-x_{N,k_N})}{|\sum_{j=1}^N\exp(\frac{2\pi i(N-1)(j-1)}{N})h_j(x_{1,k_1})|}.\end{aligned}$$

We have used $(h_l(x) = \frac{\partial X_l(x_1,\ldots,x_N)}{\partial x_l}|_{x_1=\cdots=x_N}, l=1,2,\ldots,N)$. For a given root x_{1,k_1}, the last term of sum is reduced to:

$$\sum_{k_2}\frac{\delta(x_{1,k_1}-x_{2,k_2})}{|\sum_{j=1}^{N}\exp(\frac{2\pi i(j-1)}{N})h_j(x_{1,k_1})|}\times\sum_{k_N}\frac{\delta(x_{1,k}-x_{N,k_N})}{|\sum_{j=1}^{N}\exp(\frac{2\pi i(N-1)(j-1)}{N})h_j(x_{1,k_1})|}$$

$$=\sum_{k_2,\dots,k_N}\prod_{J=2}^{N}\delta(x_{1,k_1}-x_{J,k_J})\left|\det\begin{pmatrix}h_3-h_2 & h_4-h_2 & \cdots & h_1-h_2\\ \vdots & & & \\ h_1-h_N & h_2-h_N & \cdots & h_{N-1}-h_N\end{pmatrix}\right|^{-1}$$

$$=\sum_{k_2,\dots,k_N}\prod_{J=2}^{N}\delta(f(x_{1,k_1},x_{2,k_2},\dots,x_{N,k_N})-f(x_{J,k_J},x_{J+1,k_{J+1}},\dots,x_{J-1,k_{J-1}}))$$

$$=\prod_{J=2}^{N}\delta(F(x_1,x_2,\dots,x_N)-F(x_J,x_{J+1},\dots,x_{J-1}))=\prod_{i=2}^{N}\delta(X_1-X_i),$$

where two last equalities follow from the fact $(x_{1,k},\dots,x_{N,k})\in\Phi^{-1}_{\text{coupled}}(x_1,\dots,x_N)$, i.e., $x_{i,k}$ is one of the roots of the map (1) for a given set of $\{X_1,\dots,X_N\}$ and the expansion of $\delta(F(x_{1,k},\dots,x_{N,k})-F(x_{2,k},\dots,x_{1,k}))$(by considering $x_{i,k}$, as the variable). Substituting the obtained results in (3), we obtain:

$$\begin{aligned}\mu(X_1,\dots,X_N)&=\delta(X_2-X_1)\cdots\delta(X_N-X_1)\mu(X)\\&=\delta(X_2-X_1)\cdots\delta(X_N-X_1)\\&\quad\times\sum_{(x_{1,k},\dots,x_{N,k})\in\Phi^{-1}(X_1,\dots,X_N)}\frac{\mu(x_{1,k},\dots,x_{N,k})}{|\sum_{j=1}^{N}h_j(x_{1,k},\dots,x_{N,k})|}\end{aligned}$$

which implies $\mu(X)$ to satisfy:

$$\begin{aligned}\mu(X)&=\sum_{(x_{1,k},\dots,x_{N,k})\in\Phi^{-1}(X_1,\dots,X_N)}\frac{\mu(x_{1,k},\dots,x_{N,k})}{|\sum_{j=1}^{N}h_j(x_{1,k},\dots,x_{N,k})|}\\&=\sum_{x_k\in\Phi^{-1}(X,X,\dots,X)}\frac{\mu(x_k)}{\sum_{j=1}^{N}h_j(x_k)}\end{aligned}\tag{6}$$

which is the same as PF equation of one-dimensional map $X=F(x,x,\dots,x)$. The required condition for presenting of the invariant measure of the synchronized coupled map is to choose a one-dimensional map with an invariant measure as was introduced in our previous work [16, 17].

3.2 Entropy

Entropy sometimes is called "uncertainty" and sometimes "information". In order to study the stability, entropy can be used as an acceptable parameter. Also, by considering SRB measure, it is possible to continue this study with KS-entropy, the well-known measure for chaos in dynamical systems. This section introduces the KS-entropy for N-coupled maps with a dynamical parameter. We try to calculate Lyapunov exponent as another tool to study the stability.

3.2.1 Kolmogorov–Sinai Entropy in Synchronized State

KS-entropy or metric entropy measures how chaotic a dynamical system is and proportional to the rate at which information about the state of system is lost in the course of time or iteration [18]. Using the invariant measure, the [KS-entropy] of symmetric N-dimensional map can be written as [16, 19]:

$$h(\mu,\Phi)=\int dx_1\cdots\int dx_N\mu(x_1,\ldots,x_N)\ln\left|\frac{\partial(X_1,\ldots,X_N)}{\partial(x_1,\ldots,x_N)}\right|$$
$$=\iint\cdots\int\mu(x_1,\ldots,x_N)dx_1\cdots dx_N\times\left|\det\begin{pmatrix}h_1 & h_2 & \cdots & h_N\\ h_2 & h_3 & \cdots & h_{N-1}\\ \vdots & & & \\ h_N & h_{N-1} & \cdots & h_1\end{pmatrix}\right|.$$

At synchronized state we have:

$$h(\mu,\Phi\text{ syn})=\int\cdots\int\delta(x_2-x_1)\delta(x_N-x_1)\mu(x_1)dx_1\cdots dx_N,$$

$$\left|\det\begin{pmatrix}h_1 & h_2 & \ldots & h_N\\ h_2 & h_3 & \ldots & h_{N-1}\\ \vdots & & & \\ h_N & h_{N-1} & \ldots & h_1\end{pmatrix}\right|=\prod_{k=1}^{N}\left(\sum_{j=1}^{N}\exp\left(\frac{2\pi i(j-1)(k-1)}{N}\right)h_j\right)$$
$$+h(\mu,X=f(x,\ldots,x))$$

where $h(\mu,X=f(x,x,\ldots,x))$ is the KS-entropy of one-dimensional map $X=f(x,x,\ldots,x)$ with invariant measure $\mu(x)$. The transition occurs from chaotic synchronization (spatial order with temporal chaos) to non-synchronized states with positive KS-entropy (spatial and temporal disorder).

3.2.2 Lyapunov Exponents in Synchronized State

The tangent exponent determines the stability properties of the attractor inside the symmetric subspace to synchronous disturbances. A chaotic system is sensitive to small changes of initial state. This tendency to amplify small perturbations is quantified by the Lyapunov exponent of the system (or several exponents in case of system in higher dimensions) [13, 20, 21]. In synchronized state, the Lyapunov exponents Λ_k of N-dimensional dynamical system described by the map (1) are defined as $\lim_{n\longrightarrow\infty}\frac{1}{n}|\lambda_k(x_1,\ldots,x_N)|$, where $\lambda_k=\sum_{k=1}^{N}h_k(x_1,\ldots,x_N)$ are eigenstates of the matrix:

$$\begin{bmatrix}\frac{\partial\overbrace{FoFo\cdots F}^{n}(x_1(0),\ldots,x_N(0))}{\partial x_1(0)} & \cdots & \frac{\partial\overbrace{FoFo\cdots oF}^{n}(x_1(0),\ldots,x_N(0))}{\partial x_N(0)}\\ \vdots & & \\ \frac{\partial\overbrace{FoFo\cdots F}^{n}(x_N(0),\ldots,x_{N-1}(0))}{\partial x_1(0)} & \cdots & \frac{\partial\overbrace{FoFo\cdots F}^{n}(x_N(0),\ldots,x_{N-1}(0))}{\partial x_N(0)}\end{bmatrix}_{|x_1(0)=\cdots=x_N(0)}$$
$$=\begin{bmatrix}\frac{\partial X_1}{\partial x_1} & \cdots & \frac{\partial X_1}{\partial x_N}\\ \vdots & & \\ \frac{\partial X_N(k)}{\partial x_N} & \cdots & \frac{\partial X_N}{\partial x_{N-1}}\end{bmatrix}_{|x_1=\cdots=x_N}$$

$$= \begin{bmatrix} h_1(x_1,\dots,x_N) & h_2(x_1,\dots,x_N) & \cdots & h_N(x_1,\dots,x_N) \\ h_2(x_1,\dots,x_N) & h_3(x_1,\dots,x_N) & \cdots & h_1(x_1,\dots,x_N) \\ \vdots & & & \\ h_N(x_1,\dots,x_N) & h_1(x_1,\dots,x_N) & \cdots & h_{N-1}(x_1,\dots,x_N) \end{bmatrix}$$

and $x_k = \overbrace{FoFo\cdots F}^{k}(x_0, x_0, \dots, x_0)$. Now, in case of ergodic one-dimensional map $X = F(x, x, \dots, x)$, the Lyapunov exponents can be written as:

$$\begin{aligned} \Lambda_k(\text{syn}) &= \lim_{n\to\infty} \frac{1}{n} \ln|\lambda_k(x_n)| = \lim_{n\to\infty} \frac{1}{n} \ln|\lambda_k(\overbrace{FoFo\cdots F}^{n}(x_{1_0}, x_{1_0}, \dots, x_{1_0}))| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \ln\left(\left|\sum_{i=1}^{N} h_i(x_{1_k}, \dots, x_{N_k})\right|\right) \\ &= \int dx\,\mu(x) \ln\left(\left|\sum_{i=1}^{N} h_i(x_{1_0}, \dots, x_{N_0})\right|\right). \end{aligned} \tag{7}$$

Finally comparing the KS-entropy with the sum of Λ_k we have:

$$h_{\text{KS}}(\Phi_{\text{syn}}) = \sum_{k=1}^{N} \Lambda_k(\Phi_{\text{syn}}).$$

Therefore, the ergodic choice of one-dimensional map $X = F(x, x, \dots, x)$ leads to the equality of KS-entropy with sum of Lyapunov exponents. Hence according to Pesin's theorem [20] (the equality of KS-entropy with the sum of positive Lyapunov exponents), the ergodicity of one-dimensional map $X = F(x, x, \dots, x)$ implies the ergodicity of symmetric N-dimensional map (1) unstable synchronized state (synchronized state is stable for negative critical exponent Λ_k), obviously the non-ergodic choice of $X = F(x, x, \dots, x)$ will lead to the non-ergodicity in synchronized state.

4 Example

By considering the introduced hierarchy of one-parameter families of ergodic maps as an N-coupled map, in this section we evaluate its invariant measure, KS-entropy and Lyapunov exponents (see Appendix):

- Invariant measure: In synchronized state $x_1 = \cdots = x_N = x$, the coupled map (2) by considering (13), reduces to:

$$X = F(x, x, \dots, x) = \bar{a}(\epsilon_1, \epsilon_2, \dots, \epsilon_N, a_1, a_2, \dots, a_N) \tan^2(N \arctan(\sqrt{x})) \tag{8}$$

  with $\bar{a}(\epsilon, a_1, a_2, \dots, a_N) = (\sum_{i=1}^{N} \epsilon_i a_i^p)^{\frac{1}{p}}$. As it is shown in Ref. [16], this map possesses the invariant measure of the following form:

$$\mu(x) = \frac{\sqrt{\beta}}{\sqrt{x}(1 + \sqrt{\beta} x)}, \tag{9}$$

provided that we choose the constant β as one of the positive roots of the following equation:

$$\bar{a}(\epsilon_1, \epsilon_2, \ldots, \epsilon_N, a_1, a_2, \ldots, a_N) = \left(\frac{\sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N \beta^{-k}}{\sum_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N \beta^{-k}} \right)^2$$

where [] means the greatest integer part. Now, by substituting the invariant measure of one-dimensional maps (9) in the relation (5), we get the following expression for the invariant measure of N-coupled map (2):

$$\mu(x_1, x_2, \ldots, x_N) = \delta(x_2 - x_1) \cdots \delta(x_2 - x_1) \frac{\sqrt{\beta}}{\sqrt{x}(1 + \sqrt{\beta} x)}. \qquad (10)$$

- Lyapunov exponents: In order to calculate the Lyapunov exponents of coupled map in synchronized state, we need to calculate the characteristic roots of the matrix:

$$\prod_{k=1}^{n-1} \begin{vmatrix} h_1(x_{1,k}, \ldots, x_{N,k}) & h_2(x_{1,k}, \ldots, x_{N,k}) & \cdots & h_N(x_{1,k}, \ldots, x_{N,k}) \\ \vdots & & & \\ h_N(x_{1,k}, \ldots, x_{N,k}) & h_1(x_{1,k}, \ldots, x_{N,k}) & \cdots & h_{N-1}(x_{1,k}, \ldots, x_{N,k}) \end{vmatrix}$$

$$= \begin{pmatrix} \epsilon_1 a_1^p & \epsilon_2 a_2^p & \cdots & \epsilon_N a_N^p \\ \vdots & & & \\ \epsilon_N a_N^p & \epsilon_1 a_1^p & \cdots & \epsilon_{N-1} a_{N-1}^p \end{pmatrix}^{n+1} \prod_{k=0}^{n} \left[\left(\sum_{k=1}^{N} \epsilon_k a_k^p \right)^{\frac{1}{p}-1} g'(x_k) \right]$$

$$= \left(F \begin{pmatrix} \sum_j \epsilon_j a_j^p & \\ & \ddots \end{pmatrix} F^{-1} \right)^{n+1} \prod_{k=0}^{n} \left[\left(\sum_{k=1}^{N} \epsilon_k a_k^p \right)^{\frac{1}{p}-1} g'(x_k) \right]$$

$$= \begin{pmatrix} \sum_j \epsilon_j a_j^p & & & \\ & \sum_j \epsilon_j \omega^{j-1} a_j^p & & \\ & & \ddots & \\ & & & \sum_j \epsilon_j \omega^{(j-1)(N-1)} a_j^p \end{pmatrix} \prod_{k=0}^{n} \left[\left(\sum_{k=1}^{N} \epsilon_k a_k^p \right)^{\frac{1}{p}-1} g'(x_k) \right]$$

where:

$$F = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \omega & \ldots & \omega_{N-1} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \omega_1^{N-1} & \ldots & \omega_{N-1}^{(N-1)} \end{pmatrix}$$

which yields:

$$\lambda_k(x_{1,k}, \ldots, x_{N,k}) = \prod_{k=0}^{n} \left[\left(\sum_{k=1}^{N} \epsilon_k a_k^p \right)^{\frac{1}{p}-1} g'(x_{1,k}, \ldots, x_{N,k}) \left(\sum_{k=1}^{N} \epsilon_k \omega^{(k-1)} a_k^p \right) \right]. \qquad (11)$$

Hence, we have:

$$\Lambda_k = \lim_{n\to\infty}\frac{1}{n}\ln|\lambda_k(x_{1,k},\dots,x_{N,k})|$$
$$= \ln|(1-\epsilon)a_1^p - \epsilon a_2^p| + \ln\left|\sum_{k=1}^{N}\epsilon_k\omega^{k-1}a_k^p\right| + \Lambda\left[\left(\sum_{k=1}^{N}\epsilon_k a_k^p\right)^{\frac{1}{p}} g'(x_{1,k},\dots,x_{N,k})\right].$$

Now, comparing the KS-entropy with sum of Lyapunov exponents, one can show that:

$$h_{\mathrm{KS}}(\Phi_{\mathrm{coupled}}) = \sum_{k=1}^{N}\Lambda_k.$$

The above equality follows from the ergodicity of one-dimensional map (13). Thus according to the Pesin's theorem [20], these maps are ergodic in unstable synchronized state.

- KS-entropy: In order to calculate KS-entropy of coupled map in synchronized state, we need to know the functions $h_k(x)$, where for particular choice of $g(x) = \tan^2(N\arctan(\sqrt{x}))$ we have:

$$h_k(x) = \epsilon_k a_k^p g'(x)\left(\sum_{i=1}^{N}\epsilon_i a_i^p\right)^{\frac{1}{p}-1},$$

with

$$g'(x) = \frac{N\sqrt{g(x)}}{\sqrt{x}(1+x)}(1+g(x)).$$

Hence the KS-entropy for one-parameter families of ergodic maps (13) takes the following form:

$$h_{\mathrm{KS}}(\Phi_{\mathrm{syn}}) = \ln|(1-\epsilon)a_1^p - \epsilon a_2^p| + \ln\left|\sum_{k=1}^{N}\epsilon_k\omega^{k-1}a_k^p\right|$$
$$+ 2h_{\mathrm{KS}}\left(X = \left(\sum_{k=1}^{N}\epsilon_k a_k^p\right)^{\frac{1}{P}}\tan^2(N\arctan\sqrt{x})\right). \quad (12)$$

Therefore, KS-entropy of coupled map in synchronized state is twice as big as the KS-entropy of one dimensional map $X = (\sum_{k=1}^{N}\epsilon_k a_k^p)^{\frac{1}{P}} g(x)$ (where, its analytic form and the details of calculation can be found in Ref. [16]) plus some constant terms depending on the parameters of coupled map.

5 Anti-Synchronization

Anti-synchronization is a phenomenon that the state vectors of synchronized systems have the same absolute values but opposite signs. Anti-synchronization is another noticeable phenomenon in periodic oscillators that has been known for quite a long time [22–24]. Obviously for odd integer p and odd-function $g(x) = -g(-x)$, we have anti-synchronization in coupled map (2), i.e, $x_n = -x_{n+1}$ (for $n = 0, 1, \dots$). With the same procedure as above, one can show that the invariant measure at anti-synchronized state take the form

$\mu(x_1,\dots,x_N)=\delta(x_2-x_1)\cdots(x_N-x_1)\mu(x_1)$, where $\mu(x)$ is the invariant measure of one-dimensional map $X=(\sum_{k=1}^{N}\epsilon_k a_k^p)^{\frac{1}{p}}g(x)$. For the choice of $g(x)=\tan^2(N\arctan\sqrt{x})$, the invariant measure becomes the same as (10) with $((1-\epsilon)a_1^p+\epsilon a_2^p)^{\frac{1}{p}}$ replaced with $((1-\epsilon)a_1^p-\epsilon a_2^p)^{\frac{1}{p}}$. Using this measure, one can calculate its KS-entropy and Lyapunov exponents at anti-synchronized state, where we only quote the results in the following:

- KS-entropy:

$$h_{\mathrm{KS}}(\Phi_{\text{anti-syn}})=\ln\left|\left(\sum_{k=1}^{N}\epsilon_k\omega^{k-1}a_k^p\right)\right|-\ln|((1-\epsilon)a_1^p-\epsilon a_2^p)|$$
$$+2h_{\mathrm{KS}}\left(X=\left(\sum_{k=1}^{N}\epsilon_k a_k^p\right)^{\frac{1}{p}}\tan^2(N\arctan\sqrt{x})\right)$$

- Lyapunov exponent:

$$\Lambda_k^-=\Lambda\left[\left(\sum_{k=1}^{N}\epsilon_k a_k^p\right)^{\frac{1}{p}}\tan^2(N\arctan\sqrt{x})\right],$$
$$\Lambda_k^+=\ln\left|\left(\sum_{k=1}^{N}\epsilon_k\omega^{k-1}a_k^p\right)\right|-\ln|(1-\epsilon)a_1^p-\epsilon a_2^p|$$
$$+\Lambda[((1-\epsilon)a_1^p-\epsilon a_2^p)^{\frac{1}{p}}\tan^2(N\arctan\sqrt{x})].$$

Obviously the coupled system is ergodic in anti-synchronized state according to Pesin's theorem [20]. Due to ergocity of one-dimensional map, we have $\Lambda^+>\Lambda^->0$ and the KS-entropy is equal to the sum of positive Lyapunov exponents.

6 Conclusion

In summary, This paper discussed high-dimensional N-coupled map spatiotemporal chaotic system and its driving synchronization behavior. In the driving synchronization, the ability of anti-synchronization is very important too. We calculated linear stability condition for N-coupled map. Simplified stability condition was obtained in term of control parameters and applied to N-coupled maps. Obviously, it would be interesting to introduce higher-dimensional maps with different kinds of symmetry of similar property, i.e., possession of invariant measures in different possible invariant sub-manifolds. We are now studying this problem and will give further results in other papers.

Appendix One-parameter family of chaotic maps

We give a brief review of one-dimensional chaotic maps which are going to be used. One-parameter families of chaotic maps $\Phi_N(x,a)$ of the interval [0, 1] with an invariant measure can be defined as the ratio of polynomials of degree N [16]:

$$\Phi(x,a)=\frac{a^2(T_N(\sqrt{x}))^2}{1+(a^2-1)(T_N(\sqrt{x}))^2}.$$

As an example, some of these maps given below:

$$\Phi_2(x,a) = \frac{a^2(2x-1)^2}{4x(1-x)+a^2(2x-1)^2},$$

$$\Phi_3(x,a) = \frac{a^2x(4x-3)^2}{a^2x(4x-3)^2+(1-x)(4x-1)^2}.$$

It is shown that these maps have interesting properties, that is, for even values of N the $\Phi(a,x)$ maps have only a fixed point attractor $x=1$ provided that their parameter belongs to the interval (N,∞) while, at $a \geq N$ they bifurcate to chaotic regime without having any period doubling or period-n-tupling scenario and remain chaotic for all $a \in (0,N)$ but for odd values of N, these maps have only fixed point attractor at $x=0$ for $a \in (\frac{1}{N}, N)$, again they bifurcate to a chaotic regime at $a \geq \frac{1}{N}$, and remain chaotic for $a \in (0, \frac{1}{N})$, finally they bifurcate at $a=N$ to have $x=1$ as fixed point attractor for all $a \in (\frac{1}{N}, \infty)$. Here in this paper we are concerned with their conjugate maps which are defined as:

$$\tilde{\Phi}_N(x,a) = h \circ \Phi_N(x,a) \circ h^{-1} = \frac{1}{a^2}\tan^2(N \arctan\sqrt{x}). \tag{13}$$

Conjugacy means that the invertible map $h(x) = \frac{1-x}{x}$ maps $I=[0,1]$ into $[0,\infty)$. In order to simplify the calculation of KS-entropy in this paper we denoted "$\tan^2(N \arctan\sqrt{x})$" with $g(x)$.

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